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Numerical Solution of the Nonlinear Magnetostatic-Field Equation in Two Dimensions[1]

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ABSTRACT

The numerical solution of the second-order, elliptic, quasi-linear, partial differential equation arising in a two-dimensional magnetostatic-field problem, where the magnetic permeability varies with the field, is considered. A set of nonlinear difference equations approximating the original differential equation is derived, and in solving a test problem, the method of nonlinear successive overrelaxation compares favorably both to Newton's method and to a commonly used method based on a small-magnetic-field approximation. The first method, as here presented, could also be used to numerically solve similar equations, such as those for Plateau's problem or for irrotational compressible fluid flow.

I. INTRODUCTION

In this paper, the numerical solution of the second-order, elliptic, quasi-linear, partial differential equation arising in two-dimensional magnetostatic field problems is discussed. The type of problems considered are those arising, for example, in the design of particle accelerators where the desired magnetic field strength is so large as to be principally in the domain of nonlinear behavior of the magnetic material. For such a problem, the usual successive-approximation methods [1] based on the technique of linearizing about small magnetic fields may be inadequate, and a technique involving the more essential nonlinear features should be used.

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A numerical method of the latter type is presented, which is based on the iterative solution (by nonlinear successive overrelaxation) of a set of nonlinear difference equations approximating the differential equation. Nonlinear successive overrelaxation was recently investigated by Ortega and Rockoff (or, as they more specifically describe it, extrapolated Gauss–Seidel–Newton iteration), and they found that the method compared favorably to other methods in solving a mildly nonlinear elliptic equation [2]. The method presented here for a quasi-linear equation is similar to those proposed by Lieberstein [3], Schechter [4], and Greenspan [5], but the approximating difference equations are set up differently. The performance of the method in numerically solving a sample problem is compared to those of the usual small-magnetic-field approach and of Newton's method. The results show that the method described here is better for solving the sample problem. The method could be used to numerically solve other problems governed by a similar quasi-linear differential equation, such as Plateau's problem, or that of irrotational compressible fluid flow.

II. Formulation

Consider a two-dimensional simply-connected region R in the x–y plane with boundary Γ. Let a current density in the z direction $\mathbf{J}(x, y) = J(x, y)\mathbf{k}$ be given in R; then the magnetic vector potential $\mathbf{A}(x, y) = A(x, y)\mathbf{k}$ satisfies

$$\nabla \cdot (\gamma \nabla A) = -4\pi J \text{ in } R, \tag{1}$$

and the magnetic field $\mathbf{B}$ is given by $\mathbf{B} = \nabla \times \mathbf{A} = (\partial A/\partial y)\mathbf{i} - (\partial A/\partial x)\mathbf{j}$. The quantity γ is the magnetic reluctivity (reciprocal of the magnetic permeability μ) of the material occupying R, and is a given function of $|\mathbf{B}|^2$. Since in two dimensions, $|\mathbf{B}|^2 = |\nabla \times \mathbf{A}|^2 = A_x^2 + A_y^2 = |\nabla A|^2$, γ is a function of $|\nabla A|^2$, so that when the differentiations in Eq. (1) are performed, the equation becomes

$$[\gamma + 2\gamma' A_x^2]A_{xx} + 4\gamma' A_x A_y A_{xy} + [\gamma + 2\gamma' A_y^2]A_{yy} = -4\pi J, \tag{1a}$$

where the prime denotes differentiation with respect to $|\nabla A|^2$, and the subscripts denote partial differentiation. The reluctivity for ideal materials satisfies

$$M \geq \gamma \geq m > 0 \quad \text{and} \quad M' \geq \gamma + 2\gamma' |\nabla A|^2 \geq m' > 0$$

and hence Eq. (1) is quasi-linear and uniformly elliptic. The boundary conditions for A are normally that A equals A_0, a constant, (no flux leakage) along a por-

tion or all of Γ, and that $\partial A/\partial n$, the normal derivative of A, is zero (symmetry condition) along the remainder of Γ.

The problem may also be formulated in variational terms. Find a function $A(x, y)$, twice differentiable in R satisfying the boundary conditions on Γ, that minimizes the integral

$$I = \int\int_R [g(|\nabla A|^2) - 8\pi JA]\, dx\, dy. \tag{2}$$

The given function $g(|\nabla A|^2)$ is proportional to the magnetostatic energy and is related to the reluctivity by

$$\gamma = dg/d(|\nabla A|^2). \tag{3}$$

Equation (1) is the Euler equation corresponding to Eq. (2).

In accelerator-design problems, R is usually divided into two regions, R_1 and R_2, by a curve Γ_1, and Eq. (1) [or Eq. (2)] holds separately for each region. The regions are characterized by different permeability functions. For region R_1, which is the region occupied by the ferromagnetic material, γ varies with $|\nabla A|^2$, whereas for region R_2, which is not occupied by ferromagnetic materials, γ is identically 1. In region R_2, Eq. (1) simplifies to the Poisson equation, and correspondingly, $g(|\nabla A|^2)$ in Eq. (2) simplifies to $|\nabla A|^2$. The appropriate matching condition along Γ_1 is that $\gamma(\partial A/\partial n)$ and A be continuous. In the usual case, one has $J \equiv 0$ in region R_1 and $J \not\equiv 0$ in region R_2.

The numerical solution of Eq. (1) inside of region R_2 presents little problem, since standard finite-difference methods for the Laplace operator can be used. The main difficulty arises in region R_1, where Eq. (1) is not linear. Methods commonly in use today for solving Eq. (1) are based upon obtaining the succession of linear approximating equations for A^{n+1}, the $(n+1)$th approximation to A,

$$\nabla \cdot (\gamma^n \nabla A^{n+1}) = -4\pi J, \tag{4}$$

where γ^n denotes γ as calculated from the nth approximation to A. Such methods, however, can be slowly converging or unstable when the range of A is such that γ differs significantly from a constant over R_1; this is the case when the current density J in region R_2 is large enough to partly saturate the magnetic material in R_1 [6].

In this paper another method of solving Eq. (1) is investigated which takes into account variations of γ with $|\nabla A|^2$. The method essentially corresponds to Newton's method, which obtains A^{n+1}, the $(n+1)$th approximation to A, by solving the equation

$$\nabla \cdot [\gamma^n \nabla (A^n + \varepsilon) + 2(\nabla A^n \cdot \nabla \varepsilon)(\gamma^n)' \nabla A^n] = -4\pi J \tag{5}$$

for the quantity ε, subject to the appropriate boundary conditions, and adding it to A^n,

$$A^{n+1} = A^n + \varepsilon.$$

Equation (5) is derived from Eq. (1) by neglecting all terms $O(\varepsilon^2)$. Notice that Eq. (4) lacks the term containing $(\gamma^n)'$ in Eq. (5). Thus, it is a special case of Eq. (5) when the term containing $(\gamma^n)'$ is negligible in comparison to the other retained terms, which is the case when the magnetic field is small and γ is nearly constant.

In the following sections, a set of nonlinear difference equations is derived to approximate Eq. (1), and methods of solution using successive approximations analogous to Eqs. (4) and (5) are compared for a sample problem.

III. Finite-Difference Equations

The finite-difference equations corresponding to Eq. (1) for a rectangular mesh are given in this section, and these equations are the only ones explicitly discussed in the remainder of the paper. The same method of solution may be applied to other mesh configurations as well [6].

Let the region R be covered with a rectangular mesh (not necessarily uniformly spaced) parallel to the x and y directions, and, to avoid the additional complications of boundary interpolation, let the lines intersect Γ and Γ_1 only at mesh points. Replace any curved portion of Γ and Γ_1 with a polgyonal one consisting of the chords joining adjacent mesh points. The region R is thus divided into rectangular mesh cells in its interior and either rectangular or right-triangular mesh cells at the boundary and interfaces, with each cell lying entirely in either R_1 or R_2 ([7], Sec. 6.3). Let the x and y mesh spacings be denoted by

$$h_i = x_i - x_{i-1} \quad \text{and} \quad k_j = y_j - y_{j-1}, \tag{6}$$

respectively. The difference equations satisfied by $A_{i,j}$, the discrete approximation to $A(x, y)$, can be derived by first considering a discrete analog to the variational form of the problem, Eq. (2). From this (Ritz method), a set of nonlinear difference equations can then be obtained corresponding to Eq. (1).

Approximate the integral in Eq. (2) by taking the integrand to be constant over each mesh cell. Then the integral is replaced by the sum

$$I \approx \sum_r [g(|\nabla A|^2) - 8\pi JA]_r \times \text{area}_r, \tag{7}$$

where the sum is taken over all cells into which R has been divided. The specific

form of each term in the sum will depend upon whether the corresponding cell is rectangular or triangular.

If the rth cell is rectangular, the appropriate value of the integrand to use is its value at the center of the cell (midpoint rule). The explicit expression for the term corresponding to cell $_{\bar{\imath},\bar{\jmath}}$ [the one with center at $(x_{i-1} + \frac{1}{2}h_{\bar{\imath}}, y_{j-1} + \frac{1}{2}k_{\bar{\jmath}})$; see Fig. 1] that is used here is

$$[g(|\nabla A|^2) - 8\pi JA]_{\bar{\imath},\bar{\jmath}} = g(|\nabla A|^2_{\bar{\imath},\bar{\jmath}}) - 8\pi J_{\bar{\imath},\bar{\jmath}} A_{\bar{\imath},\bar{\jmath}}, \tag{8}$$

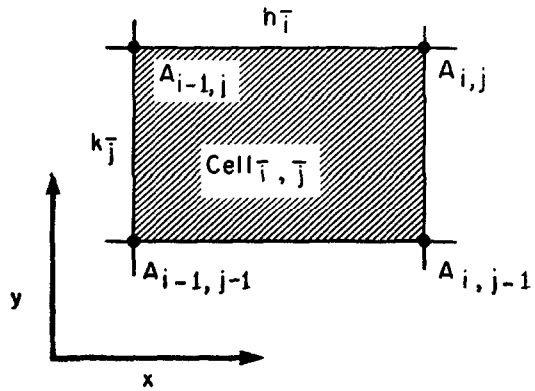

FIG. 1. General rectangular cell.

where

$$|\nabla A|^2_{\bar{\imath},\bar{\jmath}} = \frac{1}{2}\left\{\left(\frac{A_{i,j} - A_{i-1,j}}{h_{\bar{\imath}}}\right)^2 + \left(\frac{A_{i,j-1} - A_{i-1,j-1}}{h_{\bar{\imath}}}\right)^2 + \left(\frac{A_{i,j} - A_{i,j-1}}{k_{\bar{\jmath}}}\right)^2 + \left(\frac{A_{i-1,j} - A_{i-1,j-1}}{k_{\bar{\jmath}}}\right)^2\right\}, \tag{9}$$

an approximation giving $|\nabla A|^2$ at the center of the cell to $O(h_{\bar{\imath}}^2 + k_{\bar{\jmath}}^2)$,

$$A_{\bar{\imath},\bar{\jmath}} = \tfrac{1}{4}(A_{i,j} + A_{i-1,j} + A_{i,j-1} + A_{i-1,j-1}),$$

and $J_{\bar{\imath},\bar{\jmath}}$ is the given value for the average current density J crossing cell$_{\bar{\imath},\bar{\jmath}}$. The area of the cell is

$$\text{area}_{\bar{\imath},\bar{\jmath}} = h_{\bar{\imath}} k_{\bar{\jmath}}.$$

If the x and y differences of $A_{i,j}$ are denoted by

$$\delta_{\bar{\imath},j} = \frac{A_{i,j} - A_{i-1,j}}{h_{\bar{\imath}}} \quad \text{and} \quad \eta_{i,\bar{\jmath}} = \frac{A_{i,j} - A_{i,j-1}}{k_{\bar{\jmath}}},$$

then Eq. (9) can be written more simply as

$$|\nabla A|^2_{\bar{i},\bar{j}} = \tfrac{1}{2}\{\delta^2_{\bar{i},\bar{j}} + \delta^2_{\bar{i},\bar{j}-1} + \eta^2_{\bar{i},\bar{j}} + \eta^2_{\bar{i}-1,\bar{j}}\}. \tag{10}$$

If the rth cell is right-triangular, the corresponding explicit expression in Eq. (7) depends upon the orientation of the triangle. For example, for $\text{cell}_{\bar{i},\bar{j},\text{III}}$ (the one with its right angle in the third quadrant of rectangular $\text{cell}_{\bar{i},\bar{j}}$; see Fig. 2), the expression is

$$[g(|\nabla A|^2) - 8\pi JA]_{\bar{i},\bar{j},\text{III}} = g(|\nabla A|^2_{\bar{i},\bar{j},\text{III}}) - 8\pi J_{\bar{i},\bar{j},\text{III}} A_{\bar{i},\bar{j},\text{III}}, \tag{11}$$

where

$$|\nabla A|^2_{\bar{i},\bar{j},\text{III}} = \delta^2_{\bar{i},\bar{j}-1} + \eta^2_{\bar{i}-1,\bar{j}}, \tag{12}$$

$$A_{\bar{i},\bar{j},\text{III}} = \tfrac{1}{2}A_{i-1,j-1} + \tfrac{1}{4}(A_{i-1,j} + A_{i,j-1}), \tag{13}$$

and $J_{\bar{i},\bar{j},\text{III}}$ is the given value for the average current density crossing $\text{cell}_{\bar{i},\bar{j},\text{III}}$. The area of the cell is

$$\text{area}_{\bar{i},\bar{j},\text{III}} = \tfrac{1}{2}(h_{\bar{i}}k_{\bar{j}}).$$

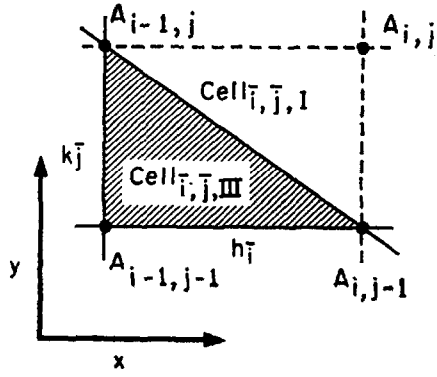

FIG. 2. Triangular cell.

The choice for $A_{\bar{i},\bar{j},\text{III}}$ is made so that the formulas for the triangular and rectangular regions are consistent. The formula for $|\nabla A|^2$ for the triangle is in general only first order, however, since it is determined by three, rather than four values of A.

The difference equations corresponding to Eq. (1) are then obtained by requiring that the partial deriative of I with respect to each of the unknown values of A_{ij} in Eq. (7) be zero. The resulting equation obtained for a general interior mesh point surrounded by four rectangles is

$$f_{ij} = \gamma_{\bar{i}\bar{j}}(\delta_{\bar{i}\bar{j}}k_{\bar{j}} + \eta_{ij}h_{\bar{i}}) + \gamma_{\bar{i}+1,\bar{j}}(-\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}} + \eta_{i,\bar{j}}h_{\bar{i}+1})$$
$$+ \gamma_{\bar{i},\bar{j}+1}(\delta_{\bar{i},\bar{j}}k_{\bar{j}+1} - \eta_{i,\bar{j}+1}h_{\bar{i}}) + \gamma_{\bar{i}+1,\bar{j}+1}(-\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}+1} - \eta_{i,\bar{j}+1}h_{\bar{i}+1})$$
$$- 2\pi(J_{\bar{i},\bar{j}}h_{\bar{i}}k_{\bar{j}} + J_{\bar{i}+1,\bar{j}}h_{\bar{i}+1}k_{\bar{j}} + J_{\bar{i},\bar{j}+1}h_{\bar{i}}k_{\bar{j}+1} + J_{\bar{i}+1,\bar{j}+1}h_{\bar{i}+1}k_{\bar{j}+1}) = 0, \qquad (14)$$

where Eq. (3) was used to substitute for $dg/d(|\nabla A|^2)$. Here $\gamma_{\bar{i}\bar{j}}$ denotes the reluctivity evaluated for $\text{cell}_{\bar{i}\bar{j}}$ with the use of Eq. (10). Equation (14) is, in general, a nonlinear one relating each A_{ij} to its eight neighbors, $A_{i\pm1,j\pm1}$, $A_{i,j\pm1}$, and $A_{i\pm1,j}$. When γ is a constant, the equation reduces to that derived from the usual five-point difference approximation to the Poisson equation.

For points along an interface or boundary bordered by triangular regions, the resulting equations are slightly more complex. For example, the equation for A_{ij} in Fig. 3 is

$$f_{ij} = \gamma_{\bar{i}\bar{j}}(\delta_{\bar{i}\bar{j}}k_{\bar{j}} + \eta_{ij}h_{\bar{i}}) + \gamma_{\bar{i}+1,\bar{j},\mathrm{III}}\eta_{ij}h_{\bar{i}+1} - \gamma_{\bar{i}+1,\bar{j},\mathrm{I}}\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}}$$
$$+ \gamma_{\bar{i},\bar{j}+1,\mathrm{III}}\delta_{\bar{i},\bar{j}}k_{\bar{j}+1} - \gamma_{\bar{i},\bar{j}+1,\mathrm{I}}\eta_{i,\bar{j}+1}h_{\bar{i}} + \gamma_{\bar{i}+1,\bar{j}+1}(-\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}+1} - \eta_{i,\bar{j}+1}h_{\bar{i}+1})$$
$$- 2\pi(J_{\bar{i},\bar{j}}h_{\bar{i}}k_{\bar{j}} + \tfrac{1}{2}[J_{\bar{i}+1,\bar{j},\mathrm{III}} + J_{\bar{i}+1,\bar{j},\mathrm{I}}]h_{\bar{i}+1}k_{\bar{j}} + \tfrac{1}{2}[J_{\bar{i},\bar{j}+1,\mathrm{III}} + J_{\bar{i},\bar{j}+1,\mathrm{I}}]$$
$$\times h_{\bar{i}}k_{\bar{j}+1} + J_{\bar{i}+1,\bar{j}+1}h_{\bar{i}+1}k_{\bar{j}+1}) = 0. \qquad (15)$$

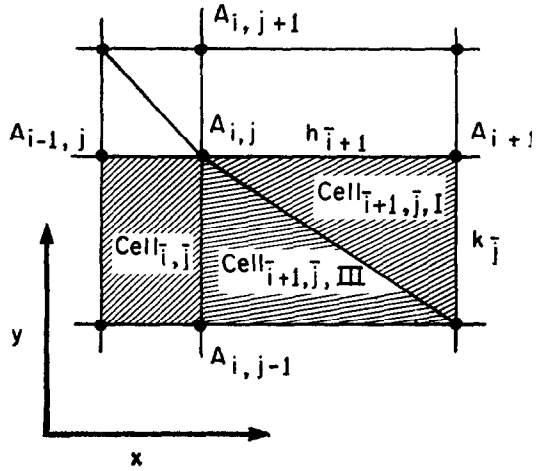

FIG. 3. Interface bordered with triangular cells.

This equation also relates A_{ij} to its eight neighbors, and reduces to the usual five-point formula when γ is a constant.

For boundary points along which $\partial A/\partial n = 0$, the obtained finite-difference equations automatically correspond to this boundary condition, because it is

the natural one for the variational problem. For boundary points along which $A = A_0$ (a constant), an additional nonlinear finite-difference equation may be obtained for A_0 by considering it to be unknown. This equation would correspond to the application of Ampere's law to the entire region R, and it may be used during the iterative solution of the problem to improve convergence by adjusting A_0 to correspond to the current approximate solution for A [8].

Because Eq. (10) or Eq. (12) was used to approximate $|\nabla A|^2$, the same finite-difference equations as those derived above could have been derived by using the line-integral equivalent to Eq. (1) obtained by the application of Green's theorem (in this case, Ampere's law) to each auxiliary mesh region, and approximating the normal derivatives by central differences ([7], Sect. 6.4). This method would be equivalent to the variational one used above, and in some cases may be algebraically more convenient. The main features to note here are that γ is a function of the unknown A values, making Eq. (14) and alterations such as Eq. (15) nonlinear in general, and that $|\nabla A|^2$ is given by Eq. (10) or (12). The Jacobian matrix of the difference equations is symmetric and although, in general, it is not diagonally dominant, its positive-definiteness follows from Schechter's arguments ([4], Sec. 9) when they are applied to the differencing scheme used here.

IV. Solution of Difference Equations

The task of solving the simultaneous nonlinear difference equations—Eq. (14) for general interior points, and possible alterations such as Eq. (15) for points near interfaces and boundaries—is approached by the commonly used small-magnetic-field method by taking γ to be a known function at each iteration, as calculated from A at previous iterations,

$$\gamma_{\bar{i},\bar{j}}^{k} = \gamma_{\bar{i},\bar{j}}^{k-1} - \omega_1[\gamma_{\bar{i},\bar{j}}^{k-1} - \gamma(|\nabla A^k|^2_{\bar{i},\bar{j}})]. \tag{16}$$

This method, which is the discrete analog to Eq. (4), then solves (or approximately solves) the resulting set of linear equations to obtain the next approximation for A. The comparison methods of interest here are based essentially on Newton's method, the discrete analog to Eq. (5), which linearizes the equations taking into account the dependence of γ on the unknown A values.

Newton's method for solving $f_{ij} = 0$ gives A_{ij}^{k+1}, the $(k+1)$th approximation to A_{ij}, as

$$A_{ij}^{k+1} = A_{ij}^{k} + \varepsilon_{ij}^{k},$$

where the ε_{ij}^k satisfy the set of linear equations

$$\sum_{l,m} (\partial f_{ij}/\partial A_{lm})^k \varepsilon_{lm}^k = -f_{ij}^k. \tag{17}$$

Equation (17) is a nine-point difference approximation to Eq. (5), having a coefficient matrix that is positive-definite, symmetric, and block-tridiagonal, each block of which is itself tridiagonal. It need be solved only approximately at each step before computing the next Newton's iterate.

The computational scheme of special interest is that of nonlinear successive overrelaxation, which is [2]–[5]

$$A_{ij}^{k+1} = A_{ij}^k - \omega \frac{f_{ij}[A_{11}^{k+1}, \cdots, A_{i-1,j}^{k+1} A_{ij}^k, \cdots, A_{n,n-1}^k]}{\partial f_{ij}/\partial A_{ij}[A_{11}^{k+1}, \cdots, A_{i-1,j}^{k+1}, A_{ij}^k, \cdots, A_{n,n-1}^k]}, \tag{18}$$

and is equivalent to performing one sweep with successive point overrelaxation on Eq. (17) when the second derivatives of f_{ij} do not vary much. Notice that only the diagonal coefficients of Eq. (17) need be computed with this scheme, but that these and f must be updated each time a new A value is calculated. The method is equivalent to performing one Newton's iteration on each equation $f_{ij} = 0$, successively, considering A_{ij} to be the only unknown and using the latest available values for the other values of A. Kronrod has suggested a variation of the method, in which one Steffensen's iteration rather than one Newton's iteration is performed on each equation [9]; his method may be a useful substitute when $\partial f_{ij}/\partial A_{ij}$ cannot be easily calculated.

The explicit expression for $\partial f_{ij}/\partial A_{ij}$ for an interior mesh point surrounded by four rectangles is obtained by differentiating Eq. (14), and is

$$\begin{aligned}
\frac{\partial f_{ij}}{\partial A_{ij}} = {} & \left(\frac{k_{\bar{j}}}{h_{\bar{i}}} + \frac{h_{\bar{i}}}{k_{\bar{j}}}\right)\gamma_{\bar{i}\bar{j}} + \left(\frac{k_{\bar{j}}}{h_{\bar{i}+1}} + \frac{h_{\bar{i}+1}}{k_{\bar{j}}}\right)\gamma_{\bar{i}+1,\bar{j}} + \left(\frac{k_{\bar{j}+1}}{h_{\bar{i}}} + \frac{h_{\bar{i}}}{k_{\bar{j}+1}}\right)\gamma_{\bar{i},\bar{j}+1} \\
& + \left(\frac{k_{\bar{j}+1}}{h_{\bar{i}+1}} + \frac{h_{\bar{i}+1}}{k_{\bar{j}+1}}\right)\gamma_{\bar{i}+1,\bar{j}+1} + \frac{\gamma'_{\bar{i}\bar{j}}}{h_{\bar{i}}k_{\bar{j}}}(\delta_{\bar{i}\bar{j}}k_{\bar{j}} + \eta_{ij}h_{\bar{i}})^2 \\
& + \frac{\gamma'_{\bar{i}+1,\bar{j}}}{h_{\bar{i}+1}k_{\bar{j}}}(-\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}} + \eta_{ij}h_{\bar{i}+1})^2 + \frac{\gamma'_{\bar{i},\bar{j}+1}}{h_{\bar{i}}k_{\bar{j}+1}}(\delta_{\bar{i}\bar{j}}k_{\bar{j}+1} - \eta_{i,\bar{j}+1}h_{\bar{i}})^2 \\
& + \frac{\gamma'_{\bar{i}+1,\bar{j}+1}}{h_{\bar{i}+1}k_{\bar{j}+1}}(\delta_{\bar{i}+1,\bar{j}}k_{\bar{j}+1} + \eta_{i,\bar{j}+1}h_{\bar{i}+1})^2,
\end{aligned} \tag{19}$$

where $\gamma'_{\bar{i}\bar{j}}$ denotes the derivative of γ with respect to $|\nabla A|^2$ evaluated for $\text{cell}_{\bar{i}\bar{j}}$. Corresponding expressions for points along an interface or boundary bordered by triangular regions are of the same form but, in general, may contain fewer or more terms.

V. Comparison and Results

The above methods were compared for the solution of a test problem having some of the essential features encountered in the ferromagnetic region R_1 of an actual accelerator-design problem. Let the square region $0 \leq x \leq 1$, $0 \leq y \leq 1$ be entirely occupied by a ferromagnetic substance with reluctivity $\gamma(w) = (10^{-4} + w)/(1 + w)$, where $w = |\nabla A|^2$ (Fig. 4). Let the current density J be identically zero and the boundary conditions on A be that $A = 0$ for $x = 0$ and $y = 1$; $A = 0.05 \sin(\pi \frac{1}{2} x)$ for $y = 0$; and $(\partial A/\partial x) = 0$ for $x = 1$. The number 0.05 is chosen so that w is of the order of 10^{-2}, so that γ, in turn, varies significantly over the rectangle, and the ratio of the second to the first term on the left of Eq. (5) is maximized. Finally, let the region be covered with a uniform square mesh so that $h = k = 1/n$ and $x_i = ih$, $y_j = jh$.

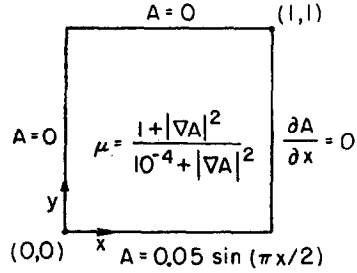

Fig. 4. Test problem.

The results of the various numerical methods for solving this test problem are shown in Table I for the case where the initial approximation to A was the solution to the linear problem (constant γ),

$$A = \frac{0.05 \sin(\pi \frac{1}{2} x) \sinh[\pi \frac{1}{2}(1 - y)]}{\sinh \frac{1}{2}\pi}.$$

The calculations were performed on the IBM 7094 by means of a FORTRAN IV program. Two meshes were considered: one containing 90 unknown points ($n = 10$) and the other containing 870 unknown points ($n = 30$). The convergence criterion in the former case was that the sum of the squares of the residuals be less than 10^{-13}, and in the latter that it be less than 10^{-12}. These correspond to an average residual at each point of the order of 10^{-6} of the maximum value of A. The iterations were ordered by letting i increase through all its values for each successively larger value of j. For each method, an optimal value of the relaxation factor was found for fastest convergence.

TABLE I

COMPARISON OF NUMERICAL METHODS

		Newton		Nonlinear successive overrelaxation	Small-field approximation
		Point	Line		
	Min./iteration	0.0011	0.0013	0.0012	0.0004
90 points	ω_{opt}	1.54	1.52	1.54	0.35 (ω_1)
Σ Res.$^2 < 10^{-13}$	Iterations to converge	21	20	18	41
	Min. to converge	0.024	0.025	0.021	0.018
	Min./iteration	0.0099	0.0112	0.0117	0.0040
870 points	ω_{opt}	1.80	1.77	1.82	0.30 (ω_1)
Σ Res.$^2 < 10^{-12}$	Iterations to converge	98	101	58	>400
	Min. to converge	0.97	1.13	0.68	>1.60

Two columns of Table I are for Newton's method [Eq. (17)]. Successive-point overrelaxation was used in the first, and successive-block overrelaxation in the second, with the relaxation factor ω. It was found that for optimal convergence it was sufficient to take only one overrelaxation sweep for each Newton iteration, signifying it was not worth solving the Newton steps too well.

Another column is for the nonlinear successive-overrelaxation method [Eq. (18)]. The iteration behaved quite stably with respect to changes in ω and in the initial approximation for A, and it was possible to achieve convergence with larger values of ω than for Newton's method, especially for the larger problem. It was found also that the centered-difference formula for $|\nabla A|^2_{\bar{i}\bar{j}}$ [Eq. (10)] gave better results than the lower-order, one-sided formulas suggested elsewhere [3]–[5].

The last column is for the small-magnetic-field approximation [Eqs. (4) and (16)], for which the iterative equations can be obtained by setting $\gamma' = 0$ in $\partial f_{ij}/\partial A_{ij}$ in Eq. (18). One Gauss–Seidel iteration ($\omega = 1$) is performed using the set of $\gamma^k_{i,j}$ calculated from Eq. (16). It was found that in order for the process to converge it is necessary to underrelax the new values of γ by choosing ω_1, the relaxation parameter in Eq. (16), to be less than one. The method did not behave as stably as the other methods with respect to changes in the initial approximation or in the relaxation parameters. Although, in some cases, it was possible to speed convergence by choosing a value of ω greater than one and a smaller value of ω_1 without the iterations diverging, for stable behavior, in general, it is necessary to use the method with ω not much greater than about one [6].

Examination of the results shows that the methods based on the nonlinear difference equations compete favorably with the small-magnetic-field approximation. Among the former methods, nonlinear successive overrelaxation performed the best. The number of iterations required for convergence was smaller than that required by Newton's method, and the total computer time used was less. The residuals decreased more uniformly during the iterations than they did with Newton's method. Nonlinear successive overrelaxation is also easier to program than is Newton's method, because only the diagonal elements of the Jacobian matrix need be computed.

Experience with the test problem has indicated, in addition, that for the centered-differencing scheme presented here, the optimal overrelaxation parameter ω for nonlinear successive overrelaxation could be estimated by using the asymptotically optimal parameter—that is, the limiting parameter that yields fastest convergence in a small neighborhood of the solution. This asymptotic parameter could, in turn, be estimated by using the relationship valid for estimating the optimal parameter for linear point successive overrelaxation applied to a matrix possessing "Property (A)" ([6]; also [10], Sec. 22.1). In fact, the entire iteration proceeded in a manner that is qualitatively the same as the behavior of such a problem. If γ is a function of x and y, the method reduces to linear point successive overrelaxation on a matrix with Property (A), but for γ a function of $|\nabla A|^2$ the Jacobian matrix, although it is block tridiagonal, does not, in general, have Property (A).

The optimal asymptotic parameter estimated from the Property (A) relationship agreed, nevertheless, with the actual observed optimal ω for the test problem to within one digit in the last of the three significant figures used. Of course, in order to obtain convergence for a poor initial approximation, it may be necessary to first use an overrelaxation parameter smaller than the optimal asymptotic one during the beginning iterations and then set ω at its optimal value later.

Parallel results for estimating the optimal relaxation parameters for the small-magnetic-field approximation could not be obtained. Thus, nonlinear successive overrelaxation recommends itself strongly over this method because, in addition to being more stable, it need not rely entirely on trial-and-error for determining optimal parameters. From these results, nonlinear successive overrelaxation shows itself to be a most promising tool for solving this type of elliptic equation.

One of the referees commented that he had been unable to obtain convergence in solving magnetostatic problems using the small-field approximation method when there were currents in the ferromagnetic region, and that it would thus be of interest to know how the method proposed here using nonlinear successive overrelaxation would work on a test problem in which $J \not\equiv 0$. Additional test

problems were run with nonzero currents and no difference was observed in the general behavior of the method. For the test problem with 90 points ($n = 10$), boundary conditions $A = 0$ on $x = 0$, $y = 0$, $y = 1$, and $\partial A/\partial x = 0$ on $x = 1$, and all currents zero except $J_{\bar{4},\bar{5}} = -J_{\bar{7},\bar{8}} = 0.002$—47 iterations are required to converge ($\sum$ Res.$^2 < 10^{-13}$) from an initial approximation of $A \equiv 0$ with $\omega = \omega_{\text{opt}} = 1.61$. The range of A and of $|\nabla A|^2$ for this problem are essentially the same as for those described in Table I. The larger number of iterations required to converge in this case is due in large part to the poorer initial approximation.

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